On a Generalization of Chatterjea's Fixed Point Theorem in b-metric Space - A Critical Study

Kshitiz Mangal Bajracharya

Abstract

Metric Spaces have been generalized in many different ways. One of those generalizations is a b-metric space. Such generalizations have resulted in generalizing some popular metric fixed point theorems in the context of a b-metric space. In 2013, Kir and Kiziltunc [6] attemped to generalize Chatterjea's Fixed Point Theorem (CFPT) in the context of b-metric spaces. The proof of that generalization, however, had a minor flaw and an unstated assumption. This paper attempts to fix these issues by introducing new conditions.

1 Preliminaries

Here, we recollect some basic concepts and definitions related to b-metric spaces.

Definition 1 (b-metric space, [1], [4], [5])**.** Let X be any non-empty set and let $b \geq 1$ be some given real number. Let $d : X \times X \longrightarrow \mathbb{R}$ be a function which satisfies the following conditions:

(b1) For all $x, y \in X$, we have $d(x, y) \geq 0$ and

$$d(x, y) = 0 \Longleftrightarrow x = y.$$

(b2) For all $x, y \in X$, we have $d(x, y) = d(y, x)$.

(b3) For all $x, y, z \in X$, we have

$$d(x, y) \leq b\,[d(x, z) + d(y, z)].$$

Then, we say that d is a b-metric defined on X and that X along with d forms a b-metric space and is denoted by the ordered pair (X, d). In some cases, if we need a distinction between b-metrics defined on different spaces, we write the space in its suffix. For example, we may write d as d_X in the above discussion. We define b as a triangular constant and refer to (b3) as relaxed triangle inequality or b-triangle inequality.

The following are examples of b-metric spaces:

Example 1. Every metric space is an example of a b-metric space because we have $b = 1$ validating the condition, (b3).

Example 2 ([2])**.** Let X be a set with three elements. Let $X = X_1 \cup X_2$ such that X_1 has two elements and $X_1 \cap X_2 = \emptyset$. Define $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & x = y, \\ 4 & x, y \in X_1 \text{ and } x \neq y, \\ 1 & x \in X_1, y \in X_2 \text{ and } x \neq y. \end{cases}$$

Then, (X, d) is a b-metric space which is not a metric space.

The following are the concepts related to sequences which we shall use in the main result.

Definition 2 (Convergent Sequence in b-metric space, [2])**.** Let (X, d) be a b-metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is said to *converge* to some $x \in X$ if for every $\varepsilon > 0$ there exists a positive integer N such that

$$n \geq N \Longrightarrow d(x_n, x) < \varepsilon.$$

This is denoted by writing

$$\lim_{n \to \infty} x_n = x.$$

Since $(d(x_n, x))_{n=1}^{\infty}$ is a sequence of positive real numbers, this definition suggests the convergence of this sequence to zero to be a characterization of convergent sequence in b-metric space. This is analogical to a similar characterization in a metric space.

Definition 3 (Cauchy Sequence in b-metric space, [2])**.** Let (X, d) be a b-metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer N such that

$$m, n \geq N \Longrightarrow d(x_m, x_n) < \varepsilon.$$

Thus, just like in the case of metric spaces, we can equivalently say that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence if $d(x_m, x_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Definition 4 (Complete b-metric Space, [2])**.** If a b-metric space (X, d) is such that every Cauchy sequence in the space is convergent, then we define it to be a complete b-metric space.

Definition 5 (Continuity of a b-metric, [4])**.** Let (X, d) be a b-metric. Then, d is said to be continuous if for any two convergent sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ of points in X, we have

$$\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$$

where, $x = \lim\limits_{n \to \infty} x_n$ and $y = \lim\limits_{n \to \infty} y_n$.

2 A Critical Study of Kir and Kiziltunc's Generalization of Chatterjea's Fixed Point Theorem (CFPT)

Kir and Kiziltunc [6] gave generalizations of Banach Fixed Point Theorem (BFPT), Kannan Fixed Point Theorem (KFPT) and Chatterjea's Fixed Point Theorem (CFPT). These theorems have been listed respectively as Theorem 1, Theorem 2 and Theorem 3 below, in the same order as they appear in [6]. The theorems have been restructured here in order to make them consistent with the notations that we have used in this paper.

Theorem 1 ([6])**.** *Let (X, d) be a complete b-metric space with a triangular constant $b > 1$. Let $T : X \longrightarrow X$ be a function for which there exists $\lambda > 0$ such that $\lambda \in (0, 1)$ and $b\lambda < 1$ which also satisfies*

$$\forall x, y \in X \quad d(Tx, Ty) \leq \lambda\, d(x, y).$$

Then, T has a unique fixed point.

Theorem 2 ([6])**.** *Let (X, d) be a complete b-metric space with a triangular constant $b > 1$. Let $T : X \longrightarrow X$ be a function for which there exists $\lambda > 0$ be such that $\lambda \in \left(0, \dfrac{1}{2}\right)$ which also satisfies*

$$\forall x, y \in X \quad d(Tx, Ty) \leq \lambda\, (d(x, Tx) + d(y, Ty)).$$

Then, T has a unique fixed point.

Theorem 3 ([6])**.** *Let (X, d) be a complete b-metric space with a triangular constant $b > 1$. Let $T : X \longrightarrow X$ be a function for which there exists $\lambda > 0$ be such that $b\lambda \in \left(0, \dfrac{1}{2}\right)$ which also satisfies*

$$\forall x, y \in X \quad d(Tx, Ty) \leq \lambda\, [d(x, Ty) + d(y, Tx)].$$

Then, T has a unique fixed point.

These theorems had one more condition, which was actually a hint to construct a Cauchy sequence for the proof, rather than a condition that was needed to construct a proof. It was to choose any $x_0 \in X$ and construct a sequence $(x_n)_{n=0}^{\infty}$ by $x_n = T^n x_0$. This sequence is then shown to be a Cauchy sequence using the conditions in the theorems. This construction has not been overlooked in this paper.

The proof of the third theorem had a flaw and the proof of $(x_n)_{n=0}^{\infty}$ being a Cauchy sequence has some unstated assumptions. We list them below.

- The flaw is that the step marked as (2.17) in their proof has been obtained by assuming the continuity of the b-metric d. The theorem doesn't state that condition and it has been illustrated by Cobzas [4] that a b-metric is not necessarily continuous.
- The proof of $(x_n)_{n=0}^{\infty}$ being a Cauchy sequence is said to be followed by using "a similar method as used in the proof of Theorem 1 and Theorem 2". Theorem 2 suggests the method similar to that of Theorem 1. So, basically, the authors want us to use the procedure as used in Theorem 1. But while doing so, we obtain

  $$d(x_m, x_n) \leq b\ k^m\ (1 + bk + (bk)^2 + \cdots + (bk)^{n-m-1})\ d(x_0, x_1).$$

  The authors have **assumed** that $bk < 1$, which lead to the conclusion that the geometric series on the right was convergent and therefore the sequence was Cauchy. Here, $k = \dfrac{b\lambda}{1 - b\lambda}$. But, what if we have $b = 20$ and $\lambda = \dfrac{1}{80}$? In such a case, we have

  $$bk = \frac{b^2\lambda}{1 - b\lambda} = \frac{\left(400 \times \dfrac{1}{80}\right)}{\left(1 - \dfrac{1}{4}\right)} = \frac{20}{3} > 1.$$

  In this case, the convergence of the said geometric sequence will not follow at all. The authors have not considered or mentioned such possibilities, which makes the proof incomplete.

Here, we wish to alter the conditions prescribed by Theorem 3 so that the new conditions would generalize CFPT to a b-metric space and has no such questionable assumptions and flaws.

3 Main Result

After critically analyzing the proof of Theorem 3, it was found that to fix the flaw of continuity of d, we need the assumption of continuity of d. And, to obtain a Cauchy sequence as we wished, it sufficed to take $bk < 1$. If $bk < 1$, then it was found that we can drop the original condition that $b\lambda \in \left(0, \dfrac{1}{2}\right)$. Luckily, the necessary "corrections" were found to be trivial. This is stated and proved formally as Theorem 4 in this paper.

Theorem 4. *Let (X, d) be a complete b-metric space with a continuous b-metric d and triangular constant $b > 1$. Let $T : X \longrightarrow X$ be a function for which there exists $\lambda > 0$ such that $0 < \dfrac{b^2\lambda}{1 - b\lambda} < 1$ and*

$$\forall x, y \in X \quad d(Tx, Ty) \leq \lambda\, [d(x, Ty) + d(y, Tx)].$$

Then, T has a unique fixed point.

Proof. Let the given condition hold. Since $0 < \dfrac{b^2\lambda}{1 - b\lambda} < 1$ and $b^2\lambda > 0$, it follows that $1 - b\lambda > 0$. Consequently, we get $0 < \dfrac{b\lambda}{1 - b\lambda} < \dfrac{b^2\lambda}{1 - b\lambda} < 1$.

To construct a Cauchy sequence, let $s \in X$ be arbitrary. Define a sequence $(x_n)_{n=1}^{\infty}$ by $x_n = T^n s$ so that, in general we get $x_{n+1} = Tx_n$. This sequence will be shown to be a Cauchy sequence. Let $n \in \mathbb{N}$.

Then

$$\begin{array}{rcll} d(x_n, x_{n+1}) & = & d(Tx_{n-1}, Tx_n) & \\ & \leq & \lambda\,[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] & \\ & = & \lambda\, d(x_{n-1}, Tx_n) & [\because x_n = Tx_{n-1}] \\ & = & \lambda\, d(x_{n-1}, x_{n+1}) & [\because x_{n+1} = Tx_n] \\ & \leq & b\,\lambda\,[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] & \end{array}$$

which implies that

$$(1 - b\lambda)\, d(x_n, x_{n+1}) \leq b\,\lambda\, d(x_{n-1}, x_n)$$

and therefore,

$$d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$$

where, $k = \dfrac{b\lambda}{1 - b\lambda}$, because $1 - b\lambda > 0$. Using this relation recursively, we get

$$d(x_n, x_{n+1}) \leq k^n\, d(s, x_1).$$

Now, let $m, n \in \mathbb{N}$ and let us take $n > m$. Then, as we have $0 < b\,k < 1$, it follows that

$$\begin{array}{rcl} d(x_m, x_n) & \leq & b\ [d(x_m, x_{m+1}) + d(x_{m+1}, n)] \\ & \leq & b\ [k^m\ d(s, x_1) + d(x_{m+1}, n)] \\ & = & b\ k^m\ d(s, x_1) + b\ d(x_{m+1}, n) \\ & \leq & b\ k^m\ d(s, x_1) + b^2\ k^{m+1} d(s, x_1) + b^2\ d(x_{m+2}, n) \\ & \vdots & \\ & \leq & b\ k^m\ (1 + bk + (bk)^2 + \cdots + (bk)^{n-m-1})\ d(s, x_1) \\ & = & b\ k^m\ \dfrac{1 - (bk)^{n-m}}{1 - (bk)}\, d(s, x_1) \\ & \leq & b\ k^m\ \dfrac{1}{1 - (bk)}\, d(s, x_1). \end{array}$$

So, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence since

$$d(x_m, x_n) \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

Thus, by completeness of X, there exists $x \in X$ such that

$$\lim_{n \to \infty} x_n = x.$$

Now, we show that x is a fixed point of T. For $n \in \mathbb{N}$, we have

$$\begin{array}{rcl} d(x, Tx) & \leq & b\,[d(x, x_{n+1}) + d(x_{n+1}, Tx)] \\ & = & b\, d(x, x_{n+1}) + b\, d(Tx_n, Tx) \\ & \leq & b\, d(x, x_{n+1}) + b\,\lambda\, d(x, Tx_n) + b\,\lambda\, d(x_n, Tx) \\ & = & b\, d(x, x_{n+1}) + b\,\lambda\, d(x, x_{n+1}) + b\,\lambda\, d(x_n, Tx). \end{array}$$

Due to the continuity of d, we get $d(x_n, Tx) \longrightarrow d(x, Tx)$ as $n \longrightarrow \infty$. So taking limits as $n \longrightarrow \infty$ in the above inequality, we get

$$d(x, Tx) \leq b\lambda\ d(x, Tx).$$

Now, as $1 - b\lambda > 0$, we have

$$\begin{array}{rcl} d(x, Tx) \leq b\lambda\ d(x, Tx) & \Longrightarrow & (1 - b\lambda)\ d(x, Tx) \leq 0 \\ & \Longrightarrow & d(x, Tx) \leq 0 \\ & \Longrightarrow & d(x, Tx) = 0. \end{array}$$

Therefore, $Tx = x$, which makes x a fixed point of T. To establish the uniqueness, let y be a different fixed point than x so that we have $y = Ty$. As $y \neq x$, we have $d(x, y > 0)$. Since x and y are fixed points of T, we have $d(x, Ty) = d(y, Tx) = d(x, y)$. So, we obtain

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\leq \lambda\,[d(x, Ty) + d(y, Tx)] \\ &= 2\lambda\, d(x, y). \end{aligned}$$

Now, as $0 < \dfrac{b\lambda}{1 - b\lambda} < 1$, it follows that $b\lambda < 1 - b\lambda$ and so, $2b\lambda < 1$. Since $b > 1$, it follows that $2\lambda < 2b\lambda < 1$. Therefore, the last inequality reduces to $d(x, y) < d(x, y)$. This is absurd. Hence, x is a unique fixed point of T. □

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